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Error Estimates for Least Squares Approximation by Polynomials

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1. THE RESULT

Let $q_0, q_1, \ldots,$ be the normalized orthogonal polynomials associated with the distribution $d\alpha$ on the fundamental interval $[-1, 1]$. The weighted least squares approximation to f is given by

$$H[f] = \sum_{\nu=0}^{n} q_\nu \int_{-1}^{1} f(t)\, q_\nu(t)\, d\alpha(t). \qquad (1)$$

The purpose of this note is to estimate the error $\|f - H[f]\|$, where $\|\cdot\|$ means the sup-norm on $[-1, 1]$. We have the

THEOREM. *Let $d\alpha$ be a distribution with the following properties*:

(i) *if f is any continuous function, then*

$$\int_{-1}^{1} f(x)\, d\alpha(x) = \int_{-1}^{1} f(-x)\, d\alpha(x),$$

(ii) $\|q_\nu\| = q_\nu(1)$, $\nu = 0, 1, \ldots, n+1$.

If the derivative $f^{(n+1)}$ exists, then

$$\|f - H[f]\| \leqslant \frac{\|q_{n+1}\|}{\|q_{n+1}^{(n+1)}\|} \|f^{(n+1)}\|. \qquad (2)$$

Remark. The example $f = q_{n+1}$ shows, that the error estimation (2) is unimprovable.

Special Case 1. $d\alpha(x) = (1 - x^2)^\beta\, dx$, $\beta \geqslant -\frac{1}{2}$. This is the expansion in terms of ultraspherical polynomials. The hypotheses of the theorem are fulfilled (Szegö [7, p. 166, p. 80]); Equation (2) is known in this case [1].

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Special Case 2. Let m be a natural number with $m \geqslant n+2$. Let $d\alpha$ be given by

$$\int_{-1}^{1} f(x)\, d\alpha(x) \equiv \frac{2}{m} \sum_{\kappa=1}^{m} f(\xi_\kappa), \qquad \xi_\kappa = -\cos \frac{2\kappa-1}{2m}\pi.$$

In this case is $q_0 = 2^{-1/2} T_0$, $q_\nu = T_\nu$ $(\nu = 1,\dots, n+1)$ (Rivlin [6, p. 49]), where T_ν denotes the Chebyshev polynomial $T_\nu(x) = \cos \nu \arccos x$. The theorem is applicable and leads to

$$\|H[f] - f\| \leqslant \frac{\|f^{(n+1)}\|}{2^n (n+1)!}. \tag{3}$$

This bound may be interesting in view of the following well-known result (Meinardus [4, p. 78]). If p denotes the polynomial of best approximation with respect to the sup-norm, then

$$\|p - f\| \leqslant \frac{\|f^{(n+1)}\|}{2^n (n+1)!}$$

is unimprovable.

The operator H makes sense for $m = n+1$, too. Then it coincides with the interpolation operator with nodes ξ_κ $(\kappa = 1,\dots, m)$. Therefore, $H[f]$ is in the general case $m > n+1$ a truncated interpolation polynomial.

Special Case 3. Let m be a natural number with $m \geqslant n+1$. Let $d\alpha$ be given by

$$\int_{-1}^{1} f(x)\, d\alpha(x) \equiv \frac{1}{m} \left\{ f(-1) + 2 \sum_{\kappa=1}^{m-1} f\left(-\cos \frac{\kappa\pi}{m}\right) + f(1) \right\}.$$

In this case we have $q_0 = 2^{-1/2} T_0$, $q_\nu = T_\nu$ $(\nu = 1,\dots, m-1)$, $q_m = 2^{-1/2} T_m$ (Rivlin [6, p. 50]). Therefore, the bound (3) holds for all $m \geqslant n+1$. This generalizes the result of Phillips and Taylor [5], who dealt with $m = n+1$. In view of results of Lewanowicz [3] the choice $m = 2n+1$ leads to a further approximation method of special interest.

2. The Proof

We require the lemma:

Lemma. *If* $|\alpha| \leqslant A$, $|\beta| \leqslant B$ *are real numbers, and* x, y *are elements of a normed space with norm* $\|\cdot\|^*$, *then*

$$\|\alpha x + \beta y\|^* \leqslant \max\{\|Ax + By\|^*, \|Ax - By\|^*\}$$

holds.

Proof. It is easily seen that $\zeta(\beta) := \|\alpha x + \beta y\|^*$ is a convex function. Therefore ζ attains its maximum on the boundary

$$\|\alpha x + \beta y\|^* \leqslant \max\{\|\alpha x + By\|^*, \|\alpha x - By\|^*\}.$$

A similar argument applied to $\psi(\alpha) := \|\alpha x \pm By\|^*$ will establish the lemma.

Proof of the Theorem. Let δ_n denote the leading coefficient of q_n. According to the Christoffel–Darboux formula we have

$$\begin{aligned} R[f] &:= f(x) - H[f](x) = H[f(x) - f](x) \\ &= \int_{-1}^{1} [f(x) - f(t)] \sum_{\nu=0}^{n} q_\nu(x)\, q_\nu(t)\, d\alpha(t) \\ &= \frac{\delta_n}{\delta_{n+1}} \int_{-1}^{1} \frac{f(x) - f(t)}{x - t} [q_{n+1}(x)\, q_n(t) - q_n(x)\, q_{n+1}(t)]\, d\alpha(t), \end{aligned}$$

if x is fixed. We define a function g by

$$g(t) = \frac{f(x) - f(t)}{x - t},$$

g has n derivatives, and the inequality

$$\|g^{(s)}\| \leqslant \frac{\|f^{(s+1)}\|}{s+1}, \qquad s = 0, 1, \ldots, n$$

holds [2]. Introducing an operator norm for functionals Q by

$$\|Q\|_s := \sup_{\|f^{(s)}\| \leqslant 1} \|Q[f]\|,$$

we have

$$\|R\|_{s+1} \leqslant \frac{\delta_n}{\delta_{n+1}} \frac{1}{s+1} \|q_{n+1}(x)\, C_n - q_n(x)\, C_{n+1}\|_s,$$
$$C_\nu[g] = \int_{-1}^{1} g(t)\, q_\nu(t)\, d\alpha(t). \tag{4}$$

As a consequence of the symmetry assumption (i) we have

$$|q_{n+1}(1)\, C_n[g] - q_n(1)\, C_{n+1}[g]| = |q_{n+1}(1)\, C_n[\bar{g}] + q_n(1)\, C_{n+1}[\bar{g}]|$$
$$\bar{g}(x) = g(-x),$$

and we conclude

$$\|q_{n+1}(1)\, C_n - q_n(1)\, C_{n+1}\|_s = \|q_{n+1}(1)\, C_n + q_n(1)\, C_{n+1}\|_s.$$

Hence, from (4) and the lemma

$$\|R\|_{s+1} \leqslant \frac{\delta_n}{\delta_{n+1}} \frac{1}{s+1} \|q_{n+1}(1)\, C_n - q_n(1)\, C_{n+1}\|_s, \tag{5}$$

$s = 0, 1,..., n$. The polynomial $q_{n+1}(1)\, q_n - q_n(1)\, q_{n+1}$ has the zero 1 and n further zeros $\eta_\nu \in [-1, 1]$ (Szegö [7, p. 45]). Let intpol$[g]$ be the interpolation polynomial for g with respect to the knots η_ν ($\nu = 1,..., n$). Using

$$g(t) - \text{intpol}[g](t) = \prod_{\nu=1}^{n} (t - \eta_\nu) \frac{g^{(n)}(\xi)}{n!}, \qquad \xi \in [-1, 1],$$

and the orthogonality, we have that

$$\begin{aligned}
&q_{n+1}(1)\, C_n[g] - q_n(1)\, C_{n+1}[g] \\
&\quad = \int_{-1}^{1} g(t)[q_{n+1}(1)\, q_n(t) - q_n(1)\, q_{n+1}(t)]\, d\alpha(t) \\
&\quad = \int_{-1}^{1} [g(t) - \text{intpol}[g](t)][\cdots]\, d\alpha(t) \\
&\quad = \frac{g^{(n)}(\xi_1)}{n!} \int_{-1}^{1} \prod_{\nu=1}^{n} (t - \eta_\nu)[\cdots]\, d\alpha(t) \\
&\quad = \frac{g^{(n)}(\xi_1)}{n!} q_{n+1}(1) \int_{-1}^{1} \prod_{\nu=1}^{n} (t - \eta_\nu)\, q_n(t)\, d\alpha(t) \\
&\quad = \frac{g^{(n)}(\xi_1)}{n!} \frac{q_{n+1}(1)}{\delta_n}.
\end{aligned}$$

This implies

$$\|q_{n+1}(1)\, C_n - q_n(1)\, C_{n+1}\|_n = \frac{q_{n+1}(1)}{n!\, \delta_n}.$$

Combining this equality with (5) concludes the proof.

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